Study Notes for Game Theory

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Contents

Chapter 1

Proof techniques for Nash Equilibrium

1.1 Fixed Point Theorem

The proof of existance of nash equilibrium is built on fixed point theorems, and there are three key fixed point theorems: Brouwer, Kakutani, and Tarski.

R^n is non-empty, compact and convex; a lattice.

1.1.1 Brouwer Fixed Point Theorem

Theorem 1 *(Brouwer Fixed Point Theorem) If $W \subseteq R^n$ is non-empty, compact and convex, then every continuous function $f : W \to W$ has a fixed point. (The proof of Brouwer Fixed Point Theorem is built on Intermediate Value Theorem by construct $g(x) = f(x) - x$ and let $g(x) = 0$)*

1.1.2 Kakutani Fixed Point Theorem

Definition 2 *(Correspondence) A **correspondence** on a set W is a function from W to the set of subsets of W. In notation, $f : W \to P(W)$. (We say x is a fixed point of $f : W \to P(W)$ iff $x \in f(x)$)*

Theorem 3 *(Kakutani Fixed Point Theorem) If $W \subseteq R^n$ is non-empty, compact and convex, and if every correspondence $f : W \to P(W)$ is non-empty valued, convex valued and has a closed graph, then f has a fixed point.*

Definition 4 *Let X is topological space, x_0 is a point in X and $f : X \to R$ is an extended real-valued function. We say f is **upper semi-continuous** at x_0 if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \leq f(x_0) + \varepsilon$ for all x in U. or equivalently, $\lim_{x \to x_0} \sup f(x) \leq f(x_0)$.*

Claim 5 *A function is upper semi-continuous if and only if $\{x \in X : f(x) < \alpha\}$ is an open set for every $\alpha \in R$.*

Definition 6 *Let X is topological space, x_0 is a point in X and $f : X \to R$ is an extended real-valued function. We say f is **lower semi-continuous** at x_0 if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0) + \varepsilon$ for all x in U. or equivalently, $\lim_{x \to x_0} \inf f(x) \geq f(x_0)$.*

Claim 7 *A function is lower semi-continuous if and only if $\{x \in X : f(x) > \alpha\}$ is an open set for every $\alpha \in R$.*

Corollary 8 *A function is continuous at x_0 iff it is upper and lower semi-continuous at x_0.*

Corollary 9 *If f and g are two real-valued functions with are both upper semi-continuous at x_0. Then $f+g$ is also upper semi-continuous at x_0.*

Corollary 10 *If f and g are two non-negative real-valued functions with are both upper semi-continuous at x_0. Then $f*g$ is also upper semi-continuous at x_0.*

Corollary 11 *If f is positive real-valued functions and upper semi-continuous at x_0. Then multiplying a negative number turns f into a lower-semi-continuous functin.*

Corollary 12 *If C is a compact space and $f : C \to [-\infty, \infty)$ is upper semi-continuous, then f has a maximum on C.*

Corollary 13 *If C is a compact space and $f : C \to (-\infty, \infty]$ is lower semi-continuous, then f has a minmum on C.*

Corollary 14 *The indicator function of any open set is lower semicontinuous.*

Corollary 15 *The indicator function of a closed set is upper semicontinuous.*

Definition 16 *A correspondence $\Gamma : A \to B$ is said to be **upper hemicontinuous** at the point a_0 if for any open neighbourhood V of $\Gamma(a_0)$ there exists a neighourhood U of a_0 such that $\Gamma(a_0)$ is a subset of V for all $x \in U$.*

Claim 17 *Upper hemicontinuity is approximately when the graph of the correspondence is closed from the left an from the right. (But not necessarily closed at every point)*

Definition 18 *A correspondence $\Gamma : A \to B$ is said to be **lower hemicontinuous** at the point a_0 if for any open set V intersecting $\Gamma(a_0)$ there exists neighbourhood U of a_0 such that $\Gamma(x)$ intersects V for all $x \in U$.*

Claim 19 *lower hemicontinuity is approximately when the graph of the correspondence has no closed edges.*

Corollary 20 *A correspondence that has both upper and lower hemicontinuous properties is said to be continuous.*

1.1.3 Tarski Fixed Point Theorem

Definition 21 *(Lattice) The partially ordered set X is a **lattice** iff every $S \subseteq X$ consisting of exactly two elements has a least upper bound and a greatest lower bound in X.*

R^n is a lattice.

Definition 22 *(Complete Lattice) The partially ordered set X is a **complete lattice** iff every $S \subseteq X$ has a least upper bound and a greatest lower bound in X.*

Claim 23 *If X is complete lattice, then for any $a, b \in X$ with $a \leq b$, the interval $\{x \in X : a \leq x \leq b\}$ is a complete lattice.*

Theorem 24 *(Tarski Fixed Point Theorem) Let X be a non-empty complete lattice. If $f : X \to X$ is weakly increasing, then the set of fixed points of f is a non-empty complete lattice.*

1.2 Proof of Existance of Equilibrium

The existance of equilibrium are usually proved by fixed point theorem. There are two way to connect NE and FPT, Fixed Point Theorem.

The first way to connect those two concepts are shown by Cachon and Netessine in chapter 2 of Handbook of quantitative supply Chain Analysis, edited by Simchi-Levi, Wu, and Shen. If the payoff functions are continuous and quasi-concave, then the best response function can be captured by FOC. So, the NE can be characterized by solving a system of best response functions, which is equalvalent to a system of FOC. E.g. if the payoff functions of 2 player game are represented by $\pi_i(x_1, x_2)$, then the best response function of each player can be represented by $\psi_1(x_2) = \arg\max_{x_1} \pi_1(x_1, x_2)$ and $\psi_2(x_1) = \arg\max_{x2} \pi_2(x_1, x_2)$. Because $\pi_i(x_1, x_2)$ are continuous and quasi-concave, FOC of the $\pi_i(x_1, x_2)$ gives the necessary condition for the optimality. Hence, best response function, $\psi_1(x_2)$ and $\psi_2(x_1)$, can be captured by FOC system $\left\{\begin{smallmatrix} x_1^*=\psi_1(x_2)\Longleftrightarrow\partial\pi_1(x_1,x_2)/\partial x_1=0 \\ x_2^*=\psi_2(x_1)\Longleftrightarrow\partial\pi_2(x_1,x_2)/\partial x_2=0 \end{smallmatrix}\right\}$. Because the NE of this game, (x_1^*, x_2^*),is solved by the intercept of two best response function, $\left\{\begin{smallmatrix} x_1^*=\psi_1(x_2^*) \\ x_2^*=\psi_2(x_1^*) \end{smallmatrix}\right.$, the NE of this game can also be captured by solving two FOC equations, $\left\{\begin{smallmatrix} \partial\pi_1(x_1,x_2)/\partial x_1|_{x_1=x_1^*,x_2=x_2^*}=0 \\ \partial\pi_2(x_1,x_2)/\partial x_2|_{x_1=x_1^*,x_2=x_2^*}=0 \end{smallmatrix}\right.$.

In order to see why we can use FPT, we can define a function $f(x_1, x_2) = \left(\begin{smallmatrix} \partial\pi_1(x_1,x_2)/\partial x_1+x_1 \\ \partial\pi_2(x_1,x_2)/\partial x_2+x_2 \end{smallmatrix}\right)$. The NE must satisfy the system equations of FOC, $\left\{\begin{smallmatrix} \partial\pi_1(x_1,x_2)/\partial x_1=0 \\ \partial\pi_2(x_1,x_2)/\partial x_2=0 \end{smallmatrix}\right.$, so at NE $f(x_1^*, x_2^*) = \left(\begin{smallmatrix} x_1^* \\ x_2^* \end{smallmatrix}\right)$, which is a fixed point.

The second way is a more directly way. First define the best response function as $\psi(x_1, x_2) = \left(\begin{smallmatrix} \psi_1(x_2) \\ \psi_2(x_1) \end{smallmatrix}\right)$. As we know the NE is defined as each player has no incentive to deviate from its current decision, we must have $\psi(x_1, x_2) = \left(\begin{smallmatrix} \psi_1(x_2) \\ \psi_2(x_1) \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ at NE. Hence, at NE, we must have $\psi(x_1^*, x_2^*) = \left(\begin{smallmatrix} \psi_1(x_2^*) \\ \psi_2(x_1^*) \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1^* \\ x_2^* \end{smallmatrix}\right)$, which is a fixed point.

1.2.1 If the payoff function are continuous and quasi-concave

Lemma 25 *If the payoff functions are continuous. Then the reaction correspondences have closed graphs. (Fudenberg and Tirole's Game Theory P31 and Thm 1.2.)*

Lemma 26 *If the payoff functions are quasi-concave in players' own actions. Then reaction correspondences are convex-valued. (Fudenberg and Tirole's Game Theory Thm 1.2.)*

Theorem 27 *(Debreu 1952). Suppose that for each player the strategy space is compact and convex and the payoff function is continuous and quasi-concave with respect to each player's own strategy. Then there exists at least one pure strategy NE in the game. (Assume the objective is to maximize the payoff function.)*

Theorem 28 *Suppose that a game is symmetric and for each player the strategy space is compact and convex and the payoff function is continuous and quasi-concave with respect each player's own strategy. Then there exists at least one symmetric pure strategy NE in the game.*

1.2.2 If the payoff functions are supermodular.

This supermodularity essentially means complementarity between any two strategies and is not linked directly to either convexity, concavity, or even continuity. So, this is very powerful when we want to work with discrete strategies.

Definition 29 *A twice continuously differentiable payoff function $\pi_i(x_1, ..., x_n)$ is* ***supermodular (submodular)*** *iff $\partial^2\pi_i/\partial x_i\partial x_j \geq 0 (\leq 0)$ for all x and all $j \neq i$. The game is called supermodular if the players" payoffs are supermodular.*[1]

[1] For more discussion of supermodular, please refer to Study Notes for Basic Mathematics.

Theorem 30 *In a supermodular game there exists at least one NE. (Use Tarski's Fixed Point Theorem)*[2]

Remark 31 *In competitive newsvendors example, the second-order cross-partial derivative, $\partial^2\pi_i/\partial Q_i \partial Q_j$, is negative, so the above theorem can not be applied. However, a standard trick is to re-define the ordering of the players' strategies by letting $y = -Q$ so that $\partial^2\pi_i/\partial Q_i \partial y$ is positive, so there exist at least one NE.*

Remark 32 *Hence, in generally, NE exists in games with decreasing best responses, submodular games, with two players.*

1.3 Proof of Uniqueness of Equilibrium

The proof of uniqueness assume the existance of equilibrium. Hence, before trying to proof the uniqueness of equilibrium, we do better first establish the existance of equilibrium.

There is no dominated method to prove the uniqueness of equilibrium. Hence, we may need to try the following 4 methods one by one to find the working one.

The uniqueness results are only available for games **with continuous best response functions** and hence there are no general methods to prove uniqueness of NE in supermodular game. [3]

However, for the proof of **uniqueness of symmetric equilibrium**, it is not too difficult. If the players have unidimensional strategies, then the system of n first-order conditions reduces to single equation and one need to show that there is a unique solution to that equation to prove the symmetric equilibrium is unique. Yet, if the players have $m-$dimensional strategies, then finding a symmetric equilibrium reduces to determining whether a system of m equations has a unique solution.

1.3.1 Method 1. Algebraic argument

- In a two-player game, the optimality condition of one of the players may have a unique closed-form solution that does not depend on other player's strategy. So, given the solution for one player, the optimality condition for the second player can be solved. If the second player has unique solution, then this problem has unique NE.
- For two-player game, one can assure uniqueness by analyzing geometrical properties of the best response function and arguling that thery intersect only once.
- For general one, we can use contradiction argument: assume that there is more than one equilibrium and prove that such an assumption leads to a contradication.

1.3.2 Method 2. Contraction mapping argument

Definition 33 *Let (X, d) be a metric space. A function $f : X \to X$ is a contraction*[4] *iff there is a number $c \in [0, 1)$ such that for any $x^1, x^2 \in X$, $d(f(x^1), f(x^2)) \leq cd(x^1, x^2)$.*

Theorem 34 *(Contraction Mapping Theorem) Let (X, d) be a non-empty complete metric space, then any contraction $f : X \to X$ has a unique fixed point.*

Theorem 35 *If the best response mapping is a contraction on the entire strategy space, then there is a unique NE in the game.*

[2] It seems the supermodular players' payoffs function gives the reaction correspondence as weakly increasing properity, so Tarski's Fixed Point Theorem applies.

[3] So, Algebraic argument, Contraction mapping argument, Univalent mapping argument, and Index theory approach are given in Cachon and Netessine in chapter 2 of Handbook of quantitative supply Chain Analysis, edited by Simchi-Levi, Wu, and Shen. So, all those four methods are require continuous best response function.

For non-continuous best response function or supermodular games, we should find out other methods to prove for uniqueness.

[4] For more rigrous definition of contraction mapping, please refer to Study Notes for Dynamic Programming.

In the theorem, the contraction mapping condition must be satisfied everywhere. However, this assumption is restrictive, because it is possible that the contraction mapping condition only holds for $\varepsilon - neiborhood$ of NE point and we have unique NE if we start within this $\varepsilon - neiborhood$. But formalization of such an argument is difficult, for some interesting discussion of stability issues in queueing system, please refer to Stidham (1992): Pricing and capacity decisions for a service facility: stability and multiple local optima.

In the following, we discuss some methods to show whether the best response function is contraction mapping. Let's $\psi_i(x_{-i})$ be the best response function for player i. Define the matrix of derivatives of best response function as

$$A = \begin{bmatrix} 0 & \partial\psi_1/\partial x_2 & ... & \partial\psi_1/\partial x_n \\ \partial\psi_2/\partial x_1 & 0 & ... & \partial\psi_2/\partial x_n \\ ... & ... & ... & ... \\ \partial\psi_n/\partial x_1 & \partial\psi_n/\partial x_2 & ... & 0 \end{bmatrix}$$

and let $\rho(A) = \{\max|\lambda| : Ax = \lambda x, x \neq 0\}$, the largest absolute eigenvalues, states the spectral radius of matrix A. Then, from Horn and Johnson 1996's Matrix Analysis and Cachon and Netessine in chapter 2 of Handbook of quantitative supply Chain Analysis, edited by Simchi-Levi, Wu, and Shen:

Theorem 36 *The mapping $\psi(x) : R^n \to R^n$ is contraction iff $\rho(A) < 1$.*

Lemma 37 *Let A be a matirx, $\rho(A)$ is its spectral radius and $\|\|$ is a consistent matrix norm. Then*

1. *for each $k \in N$: $\rho(A) \leq \left\|A^k\right\|^{1/k}, \forall k \in N$.*
2. *$\lim_{k\to\infty} A^k = 0$ iff $\rho(A) < 1$.*
3. *$\rho(A) = \lim_{k\to\infty} \left\|A^k\right\|^{1/k}$.*

Hence, the most convience way to shown $\rho(A) < 1$ is by using the above lemma: $\rho(A) \leq \|A\|$ by letting $k = 1$ and consistent norm as the maximum column-sum and maximum row-sum norms[5]. Hence, to verify the contraction mapping, it is sufficient to verify that no column sum or no row sum of matirx A exceeds one:

$$\sum\nolimits_{i=1}^{n} \left|\frac{\partial\psi_k}{\partial x_i}\right| < 1, \text{ or } \sum\nolimits_{i=1}^{n} \left|\frac{\partial\psi_i}{\partial x_k}\right| < 1, \text{ for } \forall k \tag{1.1}$$

However, sometimes the best response function can not be calculated explicitly. So, we can use implicit function theorem to simplify the row sum of matrix A in equation (1.1) as following:

$$\sum\nolimits_{i=1}^{n} \left|\frac{\partial\pi_k/\partial x_i}{\partial\pi_k/\partial x_k}\right| < 1, \text{ for } \forall k$$

which is equalvalent to

$$\sum\nolimits_{i=1}^{n} \left|\frac{\partial^2\pi_k/\partial x_i\partial x_k}{\partial^2\pi_k/\partial x_k\partial x_k}\right| < 1 \iff \sum\nolimits_{i=1}^{n} \left|\frac{\partial^2\pi_k}{\partial x_i\partial x_k}\right| < \left|\frac{\partial^2\pi_k}{\partial x_k\partial x_k}\right|, \text{ for } \forall k$$

This condition is known as **diagonal dominance**[6]: the diagonal elements of Hessian matrix has the largest absolute value within its row.

[5] This is equal to letting $k \to \infty$ and define the norm as Euclidean norm.

[6] Diagonal Dominance also imply a matrix is PSD. (Refer to Study Notes for Basic Mathematics)